Error Analysis Vade Mecum

1. Significant Figures

The precision of an experimental result is implied by the number of digits recorded.

Rules:

- The leftmost nonzero digit is the most significant digit.
- If no decimal point: the rightmost nonzero digit is the least significant digit.
- If decimal point: the rightmost digit (even if zero) is the least significant digit.
- All digits between the most and least significant digits are significant digits.

Example:

Measured value	Number of significant figures	Remarks
2	1	
2.0	2	
2.00	3	
0.136	3	
2.483	4	
2.483×10^3	4	
310	2	Ambiguous. the zero may be significant or it may only be present to show the location of the decimal point.
3.10×10^2	3	No ambiguity.
3.1×10^2	2	

Rounding:

- If the digit after the least significant digit > 5, increase the digit by one: (2.327 m ≈ 2.33 m).
- If the digit after the least significant digit < 5, keep the digit as it is: (2.323 m ≈ 2.32 m).
- If the digit after the least significant digit = 5, increase only if digit is odd (reduces systematic error due to rounding): (2.325 m ≈ 2.32 m); (2.335 m ≈ 2.34 m).

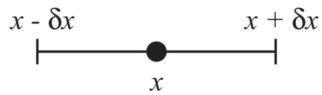
Propagation of uncertainties using significant figures:

Refer to your textbook ***Physics for Scientists and Engineers (6th edition)*** by R. A. Serway and J. W. Jewett, section 1.7 page 15.

2. Measurements

All measurements should be written as ***best estimate*** (x) ± ***uncertainty*** (δx).

$x \pm \delta x$ means that the value for the value x is probably in the range $x - \delta x$ and $x + \delta x$.



Rule for stating uncertainties:

Experimental uncertainties should almost always be rounded to one significant figure.

Rule for stating answers:

The last significant figure in any stated answer should usually be of the same order of magnitude (same decimal position) as the uncertainty.

examples:

Incorrect	Correct
8.123456 ± 0.0312 m/s	8.12 ± 0.03 m/s
$3.1234 \times 10^4 \pm 2$ m	$(3.1234 \pm 0.0002) \times 10^4$ m or 31 234 ± 2 m
$5.6789 \times 10^{-7} \pm 3 \times 10^{-9}$ kg	$(5.68 \pm 0.03) \times 10^{-7}$ kg

Absolute uncertainty:

Uncertainty expressed in the same units as the measured value.

$(75.5 \text{ g} \pm 0.5 \text{ g}) \longrightarrow \delta x = 0.5 \text{ g}$.

Fractional (relative) uncertainty:

Uncertainty expressed as a fraction of the measured value.

$$\frac{\delta x}{|x|} = \frac{0.5 \text{ g}}{75.5 \text{ g}} = 0.0066 \approx 0.007 \text{ or } 0.7\%$$

Precision:

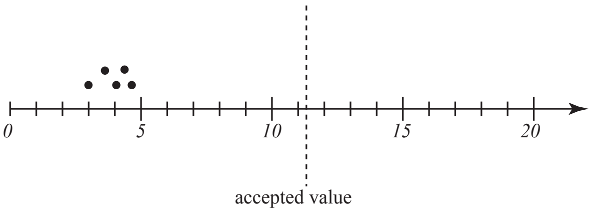
The better the precision of a measurement, the smaller the fractional uncertainty.

Accuracy:

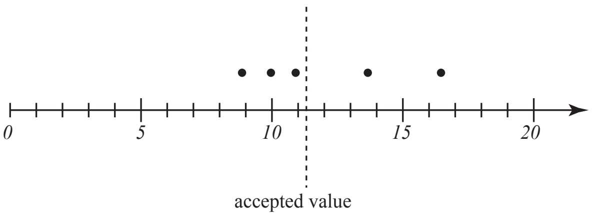
Closeness of agreement between a measured and accepted value. To talk about accuracy one must make a comparison.

Example:

The measured values (dots) in the following example are ***precise, but inaccurate***.



The measured values (dots) in the following example are ***imprecise, but accurate***.



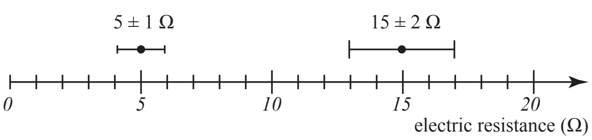
3. Comparing two measured values of the same quantity

Discrepancy:

difference between two measured values of the same quantity. Discrepancies are said to be significant or not.

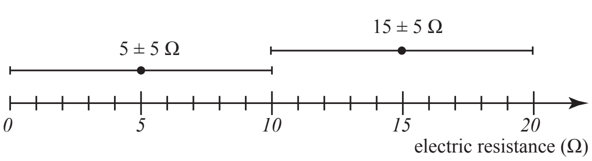
Disagreement:

In the following example, the discrepancy of 10 Ω is significant because it is much larger than the combined uncertainties of both measurements. It is said ***the two measurements are in disagreement***.

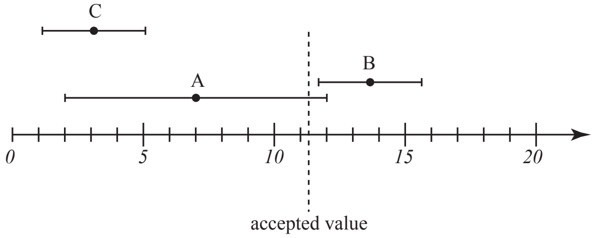


Agreement:

In the following example, the discrepancy of 10 Ω, is not significant because the uncertainties overlap. It is said ***the two measurements are in agreement***.



4. Comparing measured values with an accepted value



Measurement A agrees with the accepted value within margins of uncertainty.

Measurement B does not include the accepted value within its uncertainty range, but can nonetheless be said to be in agreement (with caution) since the discrepancy is only but slightly larger than the uncertainty.

Measurement C clearly does not agree with the accepted value, and sources of errors should be discussed.

5. Estimating Uncertainties on a single measurement

Depends on instrument used and ***how*** this instrument is used.

- precision of a graduated instrument is ***at least*** 1/2 of the smallest division, but
- calibration of the zero position,
- parallax, and
- instruments drift or fluctuations all contribute to increase the final uncertainty.

6. Estimating the uncertainty in repeated measurements

The best way to reduce random errors is to repeat the measurement *n* times.

Statistically the best estimate for these measurements is the ***average***: $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

The uncertainty in this value is given by the ***standard deviation of the mean***: $S_m = \frac{S}{\sqrt{n}}$

where S is the standard deviation defined as: $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$

The result is to be reported as $\bar{x} \pm S_m$.

Excel Functions:

Use the following functions in Excel to compute the uncertainty on repeated measurements. In this example the values would be in the cells A1 to A10.

Quantity	Function
average	=average(A1:A10)
standard deviation	=stdev(A1:A10)
standard deviation of the mean	=stdev(A1:A10)/(sqrt(count(A1:A10)))

Example:

The period of a pendulum is measured 5 times.

Period (s)	best estimate	uncertainty
14.3	14.54	0.242074369
14.9	***final result:*** 14.5 ± 0.2 s	
15.2		
14.5		
13.8		

Alternate method:

A crude estimate of the uncertainty on an average can be done very simply if you have less than 10 values. Take the range of your values and divide by the number of values : (max - min)/*n*. In the previous example, this estimate would be: $(15.2-13.8)/5 = 0.28 \simeq 0.3$

7. Propagation of uncertainties

Any quantity calculated from uncertain values will itself have an uncertainty. How do we calculate this propagation of the uncertainties. We will see several methods, from very simple to more sophisticated using calculus. In all of the descriptions below we assume the quantity q is calculated from 4 measured quantities $x \pm \delta x$, $y \pm \delta y$, $z \pm \delta z$ and $w \pm \delta w$. How do we find the uncertainty δq?

LEVEL 0

Very simple rules for beginners. These overestimates the uncertainties.

- Sums and differences → add absolute uncertainties

$$q = x + y - z - w \ \rightarrow \ \delta q = \delta x + \delta y + \delta z + \delta w$$

- Products and quotients → add fractional uncertainties

$$q = \frac{xy}{zw} \ \rightarrow \ \frac{\delta q}{|q|} = \frac{\delta x}{|x|} + \frac{\delta y}{|y|} + \frac{\delta z}{|z|} + \frac{\delta w}{|w|}$$

- Measured quantity times an exact number

$$q = Bx \ \rightarrow \ \delta q = |B|\delta x$$

- Uncertainty in a power

$$q = x^n \ \rightarrow \ \frac{\delta q}{|q|} = |n|\frac{\delta x}{|x|}$$

- Upper lower bound method for complex functions.

e.g. if an angle is measured to be $\theta \pm \delta\theta$, what is the uncertainty on $q = \cos(\theta)$? In that case q must be found in the range $q_{\min} = \cos(\theta - \delta\theta)$ and $q_{\max} = \cos(\theta + \delta\theta)$. In that case:

$$q = (q_{\min} + q_{\max})/2 \quad \text{and} \quad \delta q = |q_{\max} - q_{\min}|/2.$$

LEVEL 1

Addition in quadrature, for independent and random uncertainties this method yields a more realistic (and smaller) estimate of the final uncertainties.

- Sums and differences

$$q = x + y + z \ \rightarrow \ \delta q = \sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2}$$

- Products and quotients

$$q = \frac{xy}{zw} \ \rightarrow \ \frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{|x|}\right)^2 + \left(\frac{\delta y}{|y|}\right)^2 + \left(\frac{\delta z}{|z|}\right)^2 + \left(\frac{\delta w}{|w|}\right)^2}$$

LEVEL 2

The uncertainty on an arbitrary function of one variable *q(x)* using calculus.

$$q(x) \ \rightarrow \ \delta q = \left|\frac{dq}{dx}\right| \delta x$$

where *dq/dx* is the first derivative of q wit respect to x, and δx the absolute uncertainty on the measured value x.

Example:

suppose we have measured an angle $\theta = 20 \pm 3°$, and that we wish to find $\cos\theta$. our best estimate is of course cos 20° = 0.94, and according to the previous rule the uncertainty is:

$$\delta(\cos\theta) = \left|\frac{d\cos\theta}{d\theta}\right| \delta\theta$$

$$= |\sin\theta| \ \delta\theta(\text{in rad})$$

We have indicated that $\delta\theta$ must be expressed in radians, because the derivative of $\cos\theta$ is $-\sin\theta$ only if θ is expressed in radians. Therefore, we rewrite $\delta\theta = 3°$ as $\delta\theta$ = 0.05 rad, then

$$\delta(\cos\theta) = (\sin 20°) \times 0.05 = 0.34 \times 0.05 = 0.02$$

thus the final answer is $\cos\theta = 0.94 \pm 0.02$

LEVEL 3

General formula for error propagation. For advanced users only, with partial derivatives.

If q is a function of several variables, such as $q(x,y,z)$,

$$q(x,y,z) \;\rightarrow\; \delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2 + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

where $\partial q/\partial x$ is the partial derivative of q with respect to x, all other variables kept constant.

Example:

To determine the quantity $q = x^2y - xy^2$

A scientist measures x and y as follows: $x = 3.0 \pm 0.1$ and $y = 2.0 \pm 0.1$. What is the answer for q and its uncertainty? The best estimate for q is easily seen to be $q = 6.0$. To find δq, we follow the steps just outlined:

$$\frac{\partial q}{\partial x}\delta x = (2xy - y^2)\delta x = (12-4)\times 0.1 = 0.8$$

$$\frac{\partial q}{\partial y}\delta y = (x^2 - 2xy)\delta y = (9-12)\times 0.1 = -0.3$$

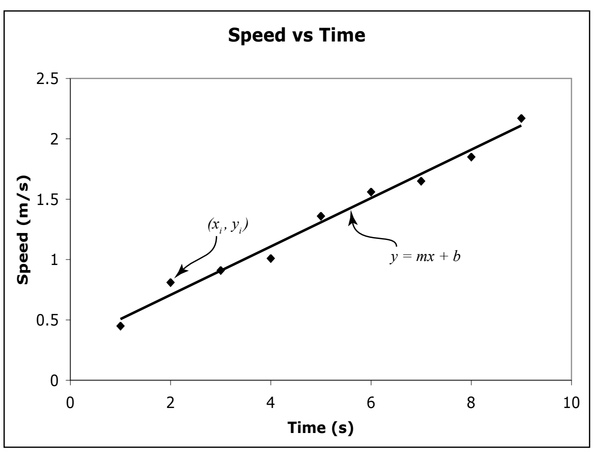
Finally the uncertainty is

$$\delta q = \sqrt{(0.8)^2 + (-0.3)^2} = 0.9$$

Thus the final answer is $q = 6.0 \pm 0.9$.

8. Graphical Analysis:

Given n data points (x_i, y_i), we want to find the equation for the "best" curve for this set of data. If the data is linearly related, then the process is called linear regression. In general, data points are not linearly related and the process of obtaining the equation for the best curve is called nonlinear regression.



Excel Functions: To make *Excel* calculate the slope, intercept and uncertainties of the best fitting line, you must use the array function called ***LINEST***. Select 4 cells in the *Excel* spreadsheet where you want the calculation to be performed. Then enter the following command

=linest(Range(y), Range(x), True, True)

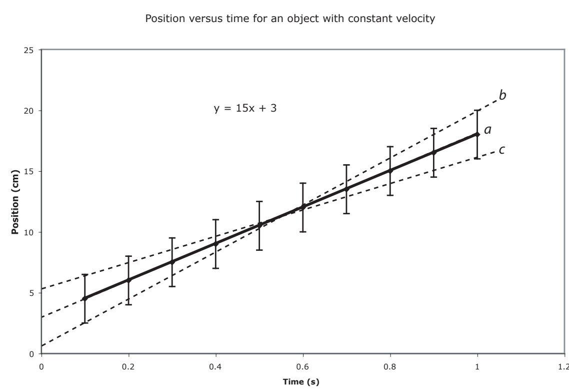
Range(y) represents the cells containing the dependent variable, and *Range(x)* the cells containing the independent variable. The two next elements are logical operator. If the first one is *True*, it means *Excel* does not force the slope through (0,0). If it was *False*, it would force it through the origin. The second *True*, tells *Excel* to calculate the uncertainties on the slope and intercept.

To evaluate an array function you must use ***CONTROL+SHIFT+ENTER***. It will then fill the four cells with the following information

Slope	intercept
uncertainty on slope	uncertainty on intercept

Alternate method (to be used in IB exams)

this method overestimates the uncertainties.



Note that three possible straight lines can be drawn in the previous graph:

- the line of best fit (best slope = m),
- the line of maximum slope (slope = m_1) and,
- the line of minimum slope (slope = m_2).

The slope, therefore, has an uncertainty calculated as $(m_1$-$m_2)/2$. The proper way to quote the value of the slope would then be $m \pm (m_1$-$m_2)/2$. Note that the lines of maximum and minimum slopes are drawn so as not to fall outside the uncertainty bars. This may not always be possible, in which case, some common sense must be applied in drawing these lines.